TRANSPORT PHENOMENA IN A PARTLY IONIZED GAS

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A closed system of transport equations for a multi component gaseous mixture was found in [1] in the "13 moment" approximation in Grad's method [2]. If the interaction between the particles (and this includes Coulomb particles) can be described in terms of collisions between pairs, an analogous system of equations can be written for ionized gas composed of an arbitrary number of uncharged (neutral) and charged components. Then, in contrast with [1], terms are introduced into the left-hand side of the equation to express the effects of the electrical and magnetic fields. A similar system of equations for a fully ionized two component plasma has been discussed recently in [3].

Normal equations of continuity, motion and energy serve as the lowest moments of the distribution function. Equations of motion for the separate gas components and the expressions for the tensors of viscous stresses and of thermal fluxes of particles, which are derived from the equation for second and third order moments, comprise a closed system which allow all the transport phenomena to be studied and the corresponding kinetic coefficients to be calculated.

In this paper expressions have been obtained for viscosity and thermal particle flux tensors in a three component plasma (electrons, ions, neutrons and neutral particles). The derivation of the generalized Ohm's Law is dealt with for such a plasma taking into account thermal particle fluxes in the equations of diffusion. Expressions are obtained for the conductivity current along and transverse to the magnetic field including the conductivity due to pressure and temperature gradients.

1. General system of equations. Significant simplification of the transport equations accrues if the assumption is made that the macroscopic gas parameters hardly change over distances of the order of the mean free path and over a time interval of the order of the collision time [1]. Assuming the particles to be at the same temperature and excluding phenomena associated with nonelastic collisions, we obtain the following general system of equations of transport for a multiform plasma within a magnetic field

$$\rho_\alpha \frac{d_\alpha \mathbf{u}_\alpha}{dt} + \operatorname{grad} p_\alpha + \operatorname{div} \boldsymbol{\pi}_\alpha - n_\alpha e_\alpha \mathbf{E}' - n_\alpha e_\alpha \mathbf{w}_\alpha \times \mathbf{B} = \tag{1.1}$$

$$= - n_\alpha \sum_\beta \mu_{\alpha\beta} \tau_{\alpha\beta}^{-1} \left[(\mathbf{w}_\alpha - \mathbf{w}_\beta) + \frac{m_\beta}{m_\alpha + m_\beta} \zeta_{\alpha\beta} \left(\frac{\mathbf{h}_\alpha}{p_\alpha} - \frac{m_\alpha}{m_\beta} \frac{\mathbf{h}_\beta}{p_\beta} \right) \right]$$

$$p_\alpha \left\{ \frac{\partial u^r}{\partial x_m} \right\} - \frac{e_\alpha}{m_\alpha} \{ \pi_\alpha{}^{lr} \varepsilon^{mlk} B^k \} = \tag{1.2}$$

$$= - n_\alpha \sum_\beta \mu_{\alpha\beta} \tau_{\alpha\beta}^{-1} \frac{kT}{m_\alpha + m_\beta} \left(a_{\alpha\beta} \frac{\pi_\alpha{}^{rm}}{p_\alpha} + a_{\alpha\beta}' \frac{\pi_\beta{}^{rm}}{p_\beta} \right)$$

$$\frac{p_\alpha}{T} \operatorname{grad} T - \frac{2}{5} \frac{e_\alpha}{kT} (\mathbf{h}_\alpha \times \mathbf{B}) + \frac{2}{5} \operatorname{div} \boldsymbol{\pi}_\alpha = \tag{1.3}$$

$$= - n_\alpha \sum_\beta \mu_{\alpha\beta} \tau_{\alpha\beta}^{-1} \left[b_{\alpha\beta} \frac{\mathbf{h}_\alpha}{p_\alpha} + b_{\alpha\beta}' \frac{\mathbf{h}_\beta}{p_\beta} + \frac{m_\beta}{m_\alpha + m_\beta} \zeta_{\alpha\beta} (\mathbf{w}_\alpha - \mathbf{w}_\beta) \right]$$

In these equations m_α, e_α, n_α, ρ_α and $\mathbf{u}_\alpha$ are, respectively, mass, charge, density, mass density and mean macroscopic velocity of particles of the α-type, p_α and $\boldsymbol{\pi}_\alpha$ are the partial pressure and the viscous stress tensor of the α-components; T is the gas temperature and k the Boltzmann constant. The relative thermal flux of the α-components $\mathbf{h}_\alpha$ and mean relative velocity $\mathbf{w}_\alpha$ are determined from the expressions

$$\mathbf{h}_\alpha = \mathbf{q}_\alpha - \tfrac{5}{2} p_\alpha \mathbf{w}_\alpha, \qquad \mathbf{w}_\alpha = \mathbf{u}_\alpha - \mathbf{u}, \qquad \mathbf{u} = \frac{1}{\rho} \sum_\alpha \rho_\alpha \mathbf{u}_\alpha \tag{1.4}$$

where $\mathbf{q}_\alpha$ is the thermal flux of particles of the α-kind and $\mathbf{u}$ the mean mass velocity of the gas. In writing down (1.1) to (1.2) the following abbreviations have been made

$$\frac{d_\alpha}{dt} = \frac{\partial}{\partial t} + (\mathbf{u}_\alpha \nabla) \qquad \{KL\}^{rm} = \tfrac{1}{2} (K^r L^m + L^r K^m) - \tfrac{1}{3} \delta^{rm} K^l L^l$$

and

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B} \tag{1.5}$$

where $\mathbf{E}$ is the electric field intensity, $\mathbf{B}$ is the vector of magnetic induction, ε^{mlk} in (1.2) denotes an "adjustable" tensor.

On the right-hand sides (1.1) to (1.3) there appear "moments of the relative collision integral" worked out with the help of the "13 moment" approximation as a function of distribution [1]. In this connection $\mu_{\alpha\beta}$ is the derived mass of particles of kind α and β, whilst the quantities $a_{\alpha\beta}$, $a_{\alpha\beta}'$, $b_{\alpha\beta}$, $b_{\alpha\beta}'$ and $\zeta_{\alpha\beta}$ depend, in general, on the mass ratios of the particles and also on the ratios of the various kinds of cross sections for a given type of collision, denoted respectively by A^*, B^* and C^*

$$a_{\alpha\beta} = 1 + 0.6\frac{m_\beta}{m_\alpha} A_{\alpha\beta}^*, \qquad a_{\alpha\beta}' = -(1 - 0.6\, A_{\alpha\beta}^*)$$

$$b_{\alpha\beta} = \frac{m_\beta^2}{(m_\alpha + m_\beta)^2}\left[1 - 0.48\, B_{\alpha\beta}^* + 0.64\frac{m_\alpha}{m_\beta} A_{\alpha\beta}^* + 1.20\left(\frac{m_\alpha}{m_\beta}\right)^2\right] \tag{1.6}$$

$$b_{\alpha\beta}' = -\frac{m_\alpha m_\beta}{(m_\alpha + m_\beta)^2}(2.20 - 0.48 B_{\alpha\beta}^* - 0.64 A_{\alpha\beta}^*)$$

$$\zeta_{\alpha\beta} = \frac{6}{5} C_{\alpha\beta}^* - 1$$

$\tau_{\alpha\beta}$ is of the same order as the time between collisions of α and β type particles. For collisions between charged and neutral particles and between neutral particles only $\tau_{\alpha\beta}$ is connected in a simple manner with the binary diffusion coefficients $[D_{\alpha\beta}]_1$ (the first Chapman-Cowling approximation [4]).

$$\tau_{\alpha\beta}^{-1} = \frac{n_\beta kT}{\mu_{\alpha\beta}\, n\, [D_{\alpha\beta}]_1} \tag{1.7}$$

In particular, for particles represented by solid elastic spheres

$$\tau_{\alpha\beta}^{-1} = \frac{16}{3} n_\beta \left(\frac{kT}{2\pi\mu_{\alpha\beta}}\right)^{1/2} Q_{\alpha\beta}, \qquad A^* = B^* = C^* = 1 \tag{1.8}$$

where $Q_{\alpha\beta}$ is the collision cross section of α and β type particles. Note that for so-called "Maxwellian" molecules

$$A^* = 5/6, \quad B^* = 3/4, \quad C^* = 5/6 \tag{1.9}$$

For the case of Coulomb interaction in working out the right-hand sides of (1.1) to (1.3) the divergence of the integrals for the collision cross-sections can be avoided by limiting the collision parameter to distances of the order of the Debye screening length λ_D. Then we have

$$\tau_{\alpha\beta}^{-1} = \frac{16}{3} n_\beta \left(\frac{2\pi kT}{\mu_{\alpha\beta}}\right)^{1/2} \left(\frac{e_\alpha e_\beta}{2kT}\right)^2 \ln \Lambda_{\alpha\beta} \tag{1.10}$$

where

$$\Lambda_{\alpha\beta} = \frac{3kT}{|e_\alpha e_\beta|}\lambda_D = \frac{3kT}{|e_\alpha e_\beta|}\left(\frac{kT}{4\pi \sum_\alpha n_\alpha e_\alpha^2}\right)^{1/2} \tag{1.11}$$

For this case

$$A_{\alpha\beta}^* = 1 - (2\ln\Lambda_{\alpha\beta})^{-1}, \qquad B^* = 1, \qquad C^* = 1/3 \tag{1.12}$$

The system of equations (1.1) to (1.3) is used for determining the viscous and thermal particle flux tensors in the plasma, and also the conductivity current

$$\mathbf{j} = \sum_\alpha n_\alpha e_\alpha \mathbf{w}_\alpha$$

First of all therefore values of τ_γ are found from (1.2) the sum of which leads to an expression for the full viscosity tensor τ. The values found should be substituted in the left-hand sides of (1.1) and (1.3). Then the values of the thermal particle fluxes $\mathbf{h}_\gamma$ are found from the solution of (1.3). The expressions obtained in this manner describe the thermal transport both due to temperature gradient and due to the relative motion of the components (diffusion). Terms proportional to div π_γ turn out to be insignificant as a rule. The total heat flux through the plasma is

$$\mathbf{q} = \sum_\alpha \mathbf{h}_\alpha + \tfrac{5}{2}\sum_\alpha p_\alpha \mathbf{w}_\alpha \tag{1.13}$$

In order to obtain diffusion velocities of the components $\mathbf{w}_\alpha$ and the conductivity current $\mathbf{j}$ associated therewith the equation of motion for the α-components (1.1) appears to be fundamental. Bearing this in mind it is convenient to bring it into a form in which the derivative $d_\alpha \mathbf{u}_\alpha/dt$ is eliminated by using the equation of gas motion in its entirety*

$$\rho \frac{d\mathbf{u}}{dt} + \operatorname{grad} p + \operatorname{div} \pi - \mathbf{j} \times \mathbf{B} = 0 \tag{1.14}$$

where
$$\rho = \sum_\alpha \rho_\alpha, \quad p = \sum_\alpha p_\alpha, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u}\nabla)$$

The result is that we obtain the following system of equations of diffusion in a multi-form plasma of diffusion in a multi-form plasma

$$-\left(\operatorname{grad} p_\alpha - \frac{\rho_\alpha}{\rho}\operatorname{grad} p\right) - \left(\operatorname{div}\pi_\alpha - \frac{\rho_\alpha}{\rho}\operatorname{div}\pi\right) +$$
$$+ n_\alpha e_\alpha \mathbf{E}' + n_\alpha e_\alpha \mathbf{w}_\alpha \times \mathbf{B} - \frac{\rho_\alpha}{\rho}\mathbf{j}\times\mathbf{B} =$$
$$= n_\alpha \sum_\beta \mu_{\alpha\beta}\tau_{\alpha\beta}^{-1}\left[(\mathbf{w}_\alpha - \mathbf{w}_\beta) + \frac{m_\beta}{m_\alpha + m_\beta}\zeta_{\alpha\beta}\left(\frac{\mathbf{h}_\alpha}{p_\alpha} - \frac{m_\alpha}{m_\beta}\frac{\mathbf{h}_\beta}{p_\beta}\right)\right] \tag{1.15}$$

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* In writing down (1.14) the condition of quasineutralness of the plasma is invoked $\sum_\alpha n_\alpha e_\alpha = 0$.

(in (1.15) the small term $(d\mathbf{u}/dt - d_\alpha \mathbf{u}_\alpha/dt)$ has been purposely omitted).

The system of diffusion equations (1.15) together with the expressions found for $\boldsymbol{\pi}_\gamma$ and $\mathbf{h}_\gamma$, and the obvious condition

$$\sum_\alpha m_\alpha n_\alpha \mathbf{w}_\alpha = 0 \tag{1.16}$$

fully determines the value of the diffusion velocities of the components and can be used for deriving the generalized Ohm's Law which connects the conductivity current $\mathbf{j}$ in the gas with the gradients of the thermodynamic quantities and the values of the electric and the magnetic fields. In what follows attention will mainly be focussed on the case of a three-component gas consisting of electrons, one kind of ion and of neutral atoms. In these discussions the mass of an ion and of a "neutral" are considered to be equal. Indeed the solution is very much simplified because of the small value of the ratio of electron mass m_e to the ion mass m_i or atomic mass m_a, in the equations. As a result, for instance, the viscous tensor and the thermal electron flux turn out to be independent of the equations of the other components, whilst the corresponding quantities for the ions and atoms are determined from the system of two equations in which $\boldsymbol{\pi}_e$ and $\mathbf{h}_e$ no longer appear.

2. Viscous tensors. Equation (1.2) for the electrons, after neglecting terms of the order $\sim m_e/m_i$, can be written down thus

$$\pi_e^{rm} = -2\eta_e e^{rm} - \tfrac{4}{3}\{\pi_e^{lr}\varepsilon^{mlk}\omega_e^k\tau_e\} \tag{2.1}$$

Here

$$\eta_e = \tfrac{2}{3} p_e\tau_e,\ \tau_e^{-1} = 0.4\tau_{ee}^{-1} + 0.8\sum_{\beta\neq e} A_{e\beta}^*\tau_{e\beta}^{-1} \tag{2.2}$$

$$\omega_e = eB/m_e \qquad (e = |e_e|) \tag{2.3}$$

$$e^{rm} = \frac{1}{2}\left(\frac{\partial u^r}{\partial x_m} + \frac{\partial u^m}{\partial x_r}\right) - \frac{1}{3}\delta^{rm}\frac{\partial u^l}{\partial x_l} \tag{2.4}$$

Solution (2.1) has the same formal appearance as that of the Chapman-Cowling solution for a charged single-component gas [4]. If $\mathbf{B}$ is directed along the x-axis the following is obtained for the viscous tensor component in Cartesian coordinates (suffix e has been dropped from the right-hand side for simplicity)

$$\pi_e^{xx} = -2\eta e^{xx}$$

$$\pi_e^{yy} = -\frac{2\eta}{1 + {}^{16}/_9\,\omega^2\tau^2}\left[e^{yy} + \frac{1}{2}(e^{yy} + e^{zz})\frac{16}{9}\omega^2\tau^2 - e^{yz}\frac{4}{3}\omega\tau\right]$$

$$\pi_e^{zz} = -\frac{2\eta}{1 + {}^{16}/_9\,\omega^2\tau^2}\left[e^{zz} + \frac{1}{2}(e^{yy} + e^{zz})\frac{16}{9}\omega^2\tau^2 + e^{yz}\frac{4}{3}\omega\tau\right]$$

$$\pi_e^{yz} = \pi_e^{zy} = -\frac{2\eta}{1+{}^{16}/_9\omega^2\tau^2}\left[e^{yz} - \frac{1}{2}(e^{zz} - e^{yy})\frac{4}{3}\omega\tau\right] \tag{2.5}$$

$$\pi_e^{xy} = \pi_e^{yx} = -\frac{2\eta}{1+{}^4/_9\,\omega^2\tau^2}\left[e^{xy} - e^{xz}\frac{2}{3}\,\omega\tau\right]$$

$$\pi_e^{xz} = \pi_e^{zx} = -\frac{2\eta}{1+{}^4/_9\,\omega^2\tau^2}\left[e^{xz} + e^{xy}\frac{2}{3}\,\omega\tau\right]$$

In these expressions $\omega = |\omega_e|$ is the cyclotron frequency of the electrons. Note that in the expressions derived by Chapman-Cowling [4] an error has crept in in writing down π_e^{yz}: preceding the second term in the square brackets the sign should be −, not +. In a worse case, as has been pointed out by Heumann, Masur and de Groot [5] the Onsager compatibility relations are violated.*

Now write down (1.2) for the ion and "neutrals" cases. Again neglecting terms of the order m_e/m_i the following system of equations appears

$$\pi_i^{rm} - \alpha\tau_i\tau_{ai}^{-1}\pi_a^{rm} = -2\eta_i e^{rm} + \frac{4}{3}\{\pi_i^{lr}\varepsilon^{mlk}\omega_i^k\tau_i\} \tag{2.6}$$

$$\pi_a^{rm} = \alpha\tau_a\tau_{ia}^{-1}\pi_i^{rm} - 2\eta_a e^{rm}$$

where

$$\eta_i = \frac{2}{3}p_i\tau_i,\quad \tau_i^{-1} = \left[0.4\tau_{ii}^{-1} + \frac{1}{3}(1+0.6A_{ia}^*)\tau_{ia}^{-1} + \frac{4}{3}\frac{m_e}{m_i}\tau_{ie}^{-1}\right] \tag{2.7}$$

$$\eta_a = \frac{2}{3}p_a\tau_a,\quad \tau_a^{-1} = \left[0.4\tau_{aa}^{-1} + \frac{1}{3}(1+0.6A_{ia}^*)\tau_{ai}^{-1} + \frac{4}{3}\frac{m_e}{m_a}\tau_{ae}^{-1}\right]$$

$$\alpha = \frac{1}{3}(1 - 0.6\,A_{ia}^*),\qquad \omega_i = \frac{ZeB}{m_i},\qquad |e_i| = Ze \tag{2.8}$$

In the expressions for τ_i^{-1} and τ_a^{-1} the last terms in the square brackets have been retained because the ratio $m_e\tau_{\beta e}^{-1}/m_\beta\tau_{\beta i}^{-1}$ $(\beta = i, a)$ is proportional to $(m_e/m_i)^{1/2}$, and not just the mass ratio alone.

On substituting π_a from the second of Equations (2.6) into the first we arrive at an equation for π_i in a form corresponding to (2.1) but incorporating different effective values of η and $\omega\tau$. Therefore the components of the viscous tensor of the ions along the coordinate axes are determined by expressions similar to (2.5) if, in the latter, the following substitutions are made

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* The author is indebted to V.B. Baranov who drew his attention to this point in [5].

$$\eta = \eta_i \frac{1 + \alpha\tau_a\tau_{ia}^{-1}}{1 - \alpha^2\tau_a\tau_i\tau_{ia}^{-1}\tau_{ai}^{-1}}, \qquad \omega\tau = -\frac{Ze|\mathbf{B}|}{m_i}\frac{\tau_i}{1 - \alpha^2\tau_a\tau_i\tau_{ia}^{-1}\tau_{ai}^{-1}} \tag{2.9}$$

The expressions for $\boldsymbol{\pi}_e$ and $\boldsymbol{\pi}_i$ agree with those found in [3] for this case for a fully ionized gas.

The viscous tensor for neutral atoms is expressed in terms of $\boldsymbol{\pi}_i$ from the second equation of (2.6). For the special case when $|\mathbf{B}| = 0$

$$\pi_a^{rm} = -2\eta_a \frac{1 + \alpha\tau_i\tau_{ai}^{-1}}{1 - \alpha^2\tau_a\tau_i\tau_{ia}^{-1}\tau_{ai}^{-1}} e^{rm} \tag{2.10}$$

The expression for the viscous tensor of the gas as a whole is obtained by adding $\boldsymbol{\pi}_e$, $\boldsymbol{\pi}_i$ and $\boldsymbol{\pi}_a$.

3. Thermal currents or fluxes. Equation (1.3) for the thermal flux of electrons, after neglecting terms of the order $\sim m_e/m_i$ can conveniently be written thus

$$\mathbf{h}_e = -\lambda_e\mathbf{R}_e - (\mathbf{h}_e \times \boldsymbol{\omega}_e\tau_e^*) \tag{3.1}$$

where

$$\lambda_e = \frac{5k}{2m_e} p_e\tau_e^*, \quad (\tau_e^*)^{-1} = 0.4\tau_{ee}^{-1} + 2.5\sum_{\beta\neq e}(1 - 0.48B_{e\beta}^*)\tau_{e\beta}^{-1} \tag{3.2}$$

$$\mathbf{R}_e = \operatorname{grad} T + \frac{2}{5}\frac{T}{p_e}\operatorname{div}\boldsymbol{\pi}_e + \frac{m_e}{k}\sum_{\beta\neq e}\tau_{e\beta}^{-1}\zeta_{e\beta}(\mathbf{w}_e - \mathbf{w}_\beta) \tag{3.3}$$

The following expression is a solution to (3.1)

$$\mathbf{h}_e = -\frac{\lambda}{1 + (\omega\tau^*)^2}[\mathbf{R} + \boldsymbol{\omega}\tau^*(\boldsymbol{\omega}\tau^*\mathbf{R}) + \mathbf{R} \times \boldsymbol{\omega}\tau^*] \tag{3.4}$$

(suffix e omitted for simplicity).

The values of $\mathbf{h}_i$ and $\mathbf{h}_a$ are determined from the system

$$\mathbf{h}_i - \beta\tau_i^*\tau_{ai}^{-1}\mathbf{h}_a = -\lambda_i\mathbf{R}_i + \mathbf{h}_i \times \boldsymbol{\omega}_i\tau_i^*, \qquad \mathbf{h}_a = \beta\tau_a^*\tau_{ia}^{-1}\mathbf{h}_i - \lambda_a\mathbf{R}_a \tag{3.5}$$

where

$$\lambda_i = \frac{5k}{2m_i} p_i\tau_i^* \tag{3.6}$$

$$(\tau_i^*)^{-1} = 0.4\tau_{ii}^{-1} + \left(\frac{11}{16} - 0.15B_{ia}^* + 0.20A_{ia}^*\right)\tau_{ia}^{-1} + 3\,\frac{m_e}{m_i}\,\tau_{ie}^{-1}$$

$$\lambda_a = \frac{5k}{2m_a}\,p_a\tau_a^* \tag{3.7}$$

$$(\tau_a^*)^{-1} = 0.4\tau_{aa}^{-1} + \left(\frac{11}{16} - 0.15B_{ia}^* + 0.20A_{ia}^*\right)\tau_{ai}^{-1} + 3\,\frac{m_e}{m_i}\,\tau_{ae}^{-1}$$

$$\mathbf{R}_i = \mathrm{grad}\,T + \frac{2}{5}\frac{T}{p_i}\,\mathrm{div}\,\pi_i + \frac{m_i}{4k}\,\zeta_{ia}\tau_{ia}^{-1}(\mathbf{w}_i - \mathbf{w}_a)$$

$$\mathbf{R}_a = \mathrm{grad}\,T + \frac{2}{5}\frac{T}{p_a}\,\mathrm{div}\,\pi_a + \frac{m_a}{4k}\,\zeta_{ai}\tau_{ai}^{-1}(\mathbf{w}_a - \mathbf{w}_i) \tag{3.8}$$

$$\beta = \frac{11}{16} - 0.15B_{ia}^* - 0.20A_{ia}^*$$

An expression for the ion thermal flux is derived from (3.5) as in the case of (3.4), by introducing these effective quantities

$$\lambda\mathbf{R} = \lambda_i\,\frac{\mathbf{R}_i + \beta\tau_a^*\tau_{ia}^{-1}\mathbf{R}_a}{1 - \beta^2\tau_i^*\tau_a^*\tau_{ai}^{-1}\tau_{ia}^{-1}}\,, \quad \omega\tau = -\frac{Ze\,|\mathbf{B}|}{m_i}\,\frac{\tau_i^*}{1 - \beta^2\tau_i^*\tau_a^*\tau_{ia}^{-1}\tau_{ai}^{-1}} \tag{3.9}$$

The thermal flux of the neutral atoms is expressed in terms of $\mathbf{h}_i$ from the second of equations (3.5). In particular when $|\mathbf{B}| = 0$

$$\mathbf{h}_a = -\,\lambda_a\,\frac{\mathbf{R}_a + \beta\tau_i^*\tau_{ai}^{-1}\mathbf{R}_i}{1 - \beta^2\tau_i^*\tau_a^*\tau_{ia}^{-1}\tau_{ai}^{-1}} \tag{3.10}$$

The total thermal flux in the gas is the sum of the corresponding partial fluxes based on (1.13). The general features of thermal transfer (transport) in directions perpendicular to and parallel to the magnetic field are similar to those described in [3,4]. In particular, a term of the form $\mathbf{R} \times \omega\tau^*$ in the expressions for the thermal particle fluxes describes the well known effects of Riggi-Lediuc and of Ettingshausen. Note that for a fully ionized gas the expressions for $\mathbf{h}_e$ and $\mathbf{h}_i$ correspond to those found in [3].

4. Generalized Ohm's Law. It has been noted in the foregoing that in deriving the generalized Ohm's Law the equations of motion (1.1) or the diffusion equations which are deduced from them (1.15) form the starting point or basis. According to the traditional method of deriving this law [6,7] (hydrodynamic approximation) the magnitude of the impulse transmitted by collisions between particles of the α kind with those of other components (right-hand side of (1.1)) is taken to be proportional to the respective differences of the macroscopic component velocities. A prominent feature of the "13 moment" approximation is the more accurate determination of this quantity as a result of which additional terms appear in the diffusion equations which are proportional to the relative thermal flux of the particles. By taking these terms into account in the normal problem of multicomponent gas mixtures (absence of charged particles) improvement in accuracy of the coefficients of mutual diffusion (second approximation) accrues and likewise in the case of thermal diffusion (further details in [1]). The significance of the additional terms is therefore relatively small in the case of real potentials of interactions between molecules ($\zeta_{\alpha\beta} \leqslant 0.2$) (for "Maxwellian" molecules $\zeta_{\alpha\beta} = 0$).

The effect of Coulomb interactions between charged particles makes a more significant contribution, when $\zeta_{\alpha\beta} = -0.6$. When thermal fluxes are taken into account in the diffusion equations markedly improved accuracy in the conductivity value σ is evident and it is also possible to consider the conductivity currents in terms of the temperature gradient.

The derivation of the generalized Ohm's Law for a three component plasma will now be discussed. Writing down (1.15) for the electron component we have

$$-\operatorname{grad} p_e - n_e e\mathbf{E}' - n_e m_e \mathbf{w}_e \times \boldsymbol{\omega}_e = n_e m_e \left[\tau_0^{-1}\mathbf{V}_e + \tau_{ea}^{-1}\mathbf{V}_i + \nu_0 \frac{\mathbf{h}_e}{p_e} \right] \tag{4.1}$$

Here

$$\mathbf{V}_e = \mathbf{w}_e - \mathbf{w}_i, \qquad \mathbf{V}_i = \mathbf{w}_i - \mathbf{w}_a \tag{4.2}$$

$$\tau_0^{-1} = \tau_{ei}^{-1} + \tau_{ea}^{-1}, \quad \nu_0 = \zeta_{ei}\tau_{ei}^{-1} + \zeta_{ea}\tau_{ea}^{-1} \tag{4.3}$$

(In (4.1) the term div $\boldsymbol{\pi}_e$ has been omitted for simplicity because in most problems connected with plasma conductivity it has little significance.) Introducing the degree of ionization

$$\alpha = \frac{n_i}{n_i + n_a} \tag{4.4}$$

and neglecting terms of the order $\sim m_e/m_i$, we have

$$\mathbf{w}_e = \frac{1}{\rho}\left[(\rho_i + \rho_a)\mathbf{V}_e + \rho_a \mathbf{V}_i\right] = \mathbf{V}_e + (1-\alpha)\mathbf{V}_i \tag{4.5}$$

It is easy to see that the expression for the thermal electron flux $\mathbf{h}_e$ (3.4) can be split into terms containing grad T, $\mathbf{V}_e$ and $\mathbf{V}_i$. Then to determine the conductivity current from (4.1)

$$\mathbf{j} = -n_e e \mathbf{V}_e \tag{4.6}$$

it is essential to invoke one additional expression which connects the "slip" velocity of the ions $\mathbf{V}_i$ with the relative velocity of the ions and electrons $\mathbf{V}_e$. To do this, Equation (1.15) written down for the ion component is used. Bearing in mind that $\rho_i/\rho \approx \alpha$ and $n_i e_i = n_e e$, we have

$$-(\operatorname{grad} p_i - \alpha \operatorname{grad} p) + n_e e_i \mathbf{E}' + n_e m_e \mathbf{w}_i \times \boldsymbol{\omega}_e + \alpha n_e m_e \mathbf{V}_e \times \boldsymbol{\omega}_e = \tag{4.7}$$
$$= -n_e m_e \tau_{ei}^{-1}\left(\mathbf{V}_e + \zeta_{ei}\frac{\mathbf{h}_e}{p_e}\right) + \frac{1}{2} n_i m_i \tau_{ia}^{-1}\mathbf{V}_i + \frac{1}{4} n_i m_i \left(\frac{\mathbf{h}_i}{p_i} - \frac{\mathbf{h}_a}{p_a}\right)\zeta_{ia}\tau_{ia}^{-1}$$

Add (4.1) and (4.7) and derive an expression for $\mathbf{V}_i$ thus

$$\mathbf{V}_i = -\frac{\varepsilon}{1+\varepsilon}\left[\mathbf{V}_e + \zeta_{ea}\frac{\mathbf{h}_e}{p_e} + (1-\alpha)\,\mathbf{V}_e \times \boldsymbol{\omega}_e \tau_{ea}\right] +$$
$$+ \frac{2\tau_{ia}}{(1+\varepsilon)\,n_i m_i}\left[\alpha \operatorname{grad} p - \operatorname{grad}(p_e + p_i)\right] - \frac{1}{2(1+\varepsilon)}\zeta_{ia}\left(\frac{\mathbf{h}_i}{p_i} - \frac{\mathbf{h}_a}{p_a}\right) \tag{4.8}$$

where

$$\varepsilon = \frac{2 n_e m_e \tau_{ea}^{-1}}{n_i m_i \tau_{ia}^{-1}} \qquad \left(\varepsilon \sim Z\left(\frac{m_e}{m_i}\right)^{1/2} \ll 1\right) \tag{4.9}$$

In order to obtain an explicit expression for $\mathbf{V}_i$ it is essential to substitute into (4.8) the expressions found above for the thermal particle flux. It is easy to see moreover that the corrections to $\mathbf{V}_i$ are proportional to ζ_{ea}^2 and ζ_{ia}^2, and amount, according to estimates, is less than 2%. A rather larger contribution is made by the terms which are proportional to the temperature gradient, but the corrections on this account in the final expression for the conductivity current are but small. Below, therefore, contributions due to $\mathbf{h}_e$, $\mathbf{h}_i$ and $\mathbf{h}_a$ are neglected in the interests of simplification in the expression for slip velocity $\mathbf{V}_i$ (for Maxwell molecules the contribution from these terms is strictly zero).

Substituting $\mathbf{V}_i$ into (4.1) taking into account (4.5) and (3.4) it should be noticed that in the expression so obtained terms proportional to the small quantity ε can be neglected everywhere when it is multiplied by $\boldsymbol{\omega}_e$ (because $\boldsymbol{\omega}_e \sim \mathbf{B}$ and $|\mathbf{B}|$ can take on arbitrary values). As a result we arrive at the following relation between conductivity current $\mathbf{j}$ and the parameters which represent the state of the gas

$$A\mathbf{j} + B\mathbf{j} \times \boldsymbol{\omega}_e \tau_0 - C\boldsymbol{\omega}_e \tau_0 (\boldsymbol{\omega}_e \tau_0 \cdot \mathbf{j}) = \sigma_0 (\mathbf{D} - \mathbf{H} \times \boldsymbol{\omega}_e \tau_0) \tag{4.10}$$

Here

$$A = 1 - \frac{\Delta_0}{1 + \gamma^2 (\omega_e \tau_0)^2} + \delta_0 (\omega_e \tau_0)^2 \qquad B = 1 + \gamma \frac{\Delta_0}{1 + \gamma^2 (\omega_e \tau_0)^2} \tag{4.11}$$
$$C = \delta_0 + \gamma^2 \frac{\Delta_0}{1 + \gamma^2 (\omega_e \tau_0)^2}$$

$$\mathbf{D} = \mathbf{E}' + \frac{1}{n_e e}\,\mathrm{grad}\,p_e - \frac{1}{1+\gamma^2(\omega_e\tau_0)^2}\alpha_T\frac{k}{e}\,[\mathrm{grad}\,T + \gamma^2\omega_e\tau_0(\omega_e\tau_0\,\mathrm{grad}\,T)] \tag{4.12}$$

$$\mathbf{H} = -\frac{\gamma}{1+\gamma^2(\omega_e\tau_0)^2}\,\alpha_T\frac{k}{e}\,\mathrm{grad}\,T + \frac{\delta_0}{(1-\alpha)n_e e}\,[\alpha\,\mathrm{grad}\,p - \mathrm{grad}\,(p_e+p_i)]$$

$$\sigma_0 = \frac{n_e e^2\tau_0}{m_e} \tag{4.13}$$

In these expressions

$$\Delta_0 = \alpha_T\nu_0\tau_0, \qquad \delta_0 = (1-\alpha)^2\,\varepsilon\tau_{ea}\,\tau_0^{-1}, \qquad \gamma = \tau_e^*\tau_0^{-1} \tag{4.14}$$

whilst the thermal diffusion constant α_T is found from

$$\alpha_T = \frac{5}{2}\nu_0\tau_e^* = \frac{5}{2}\gamma\nu_0\tau_0 \tag{4.15}$$

Equation (4.10) expresses the generalized Ohm's Law for a partly ionized gas. If we put $\alpha_T = 0$ (4.10) transforms into the well known results of the literature (see for instance Formula (2.10) of [7]. It is easy to observe, incidentally, that by considering thermal particle fluxes in the diffusion equations ($\alpha_T = 0$), not only do additional thermal diffusion terms appear but noticeable corrections accrue to the magnitude of the electrical conductivity of the plasma σ in the direction of, and transversely to, the magnetic field. Thus for a conductivity current $\mathbf{j}$ in the x direction parallel to the lines of force of the magnetic field we have

$$j_x = \sigma\left[\;E'_x + \frac{1}{n_e e}\,(\mathrm{grad}\,p_e)_x - \alpha_T\frac{k}{e}\,(\mathrm{grad}\,T)_x\right] \tag{4.16}$$

where

$$\sigma = \frac{\sigma_0}{1-\Delta_0} \tag{4.17}$$

In the case of a weakly ionized gas, when only the interaction between the electrons and neutral particles is taken into account, $\Delta_0 \approx 1.9\,\zeta_{ea}^{\;2}$, i.e. the correction in σ does not exceed 8%. In another limiting case (fully ionized gas) taking into account electron-electron and electron-ion interactions leads to $\Delta_0 = 0.484$ when $z = 1$, i.e. σ almost doubles as compared with σ_0. This result agrees with the so-called "second approximation" of Chapman-Cowling in kinetic coefficients [4] and it agrees very well with the value of σ quoted by Spitster [8].

Finally let us deal with the expression for conductivity current transverse to the magnetic field, which is derived from (4.10). Multiplying both right-hand side and left-hand side of (4.10) vectorially by

$\omega_e\tau_0$ and expanding doubly the vector product we arrive at the result

$$\mathbf{j}_\perp = \frac{\sigma_0}{A^2 + B^2(\omega_e\tau_0)^2}\,[A\mathbf{D}_\perp - (\omega_e\tau_0)^2 B\mathbf{H}_\perp - (B\mathbf{D}_\perp + A\mathbf{H}_\perp) \times \boldsymbol{\omega}_e\tau_0] \qquad (4.18)$$

where $\mathbf{D}_\perp$ and $\mathbf{H}_\perp$ are the respective vector components located in a plane perpendicular to the vector of the magnetic field intensity.

In the absence of pressure and temperature gradients the coefficient of electrical conductivity for $\mathbf{E}' \perp \mathbf{B}$ can be written down in complex form thus

$$\sigma = \sigma_0 \frac{A - i\omega_e\tau_0 B}{A^2 + B^2(\omega_e\tau_0)^2} \qquad (4.19)$$

and this agrees with the expression obtained for this particular case in [9].

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